

Low-energy theorems and spectral density of the Dirac operator in AdS/QCD

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Abstract

We study the low-energy theorems of QCD from the point of view of the dual AdS/QCD models and demonstrate that these models are compatible with the theorems in the chiral limit, i.e. the arising expressions have the same analytical behavior at the pole when the quark mass tends to zero. Low-energy theorems are formulated in terms of the spectral density of the Dirac operator. In order to calculate the spectral density in the dual holographic models we express it in terms of a partition function of a QCD-like theory.

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I. INTRODUCTION

In the recent past new approaches to studying the low-energy dynamics of QCD based on the idea of AdS/CFT correspondence have emerged. The idea of this correspondence was first put forward in [1]. The prescription of the latter consists in using a classical multidimensional string theory to describe a quantum four-dimensional field theory in a strong coupling regime. Namely, one puts into correspondence operators in the quantum field theory and multidimensional classical fields on the string theory side using their transformation properties with respect to global symmetries of the field theory and isometries of the string theory background space-time. Then the AdS/CFT conjecture states that the generating functional of the quantum field theory equals the exponent of the action of the gravity theory in which the multidimensional fields are confined to classical trajectories and their boundary values are set to be equal to the sources of the corresponding operators on the four-dimensional side:

$$\mathcal{Z}_{QFT}[J_i(x_\mu)] = \exp(iS_{Gravity \ Cl.})|_{\Phi_i(x_\mu, y_M=0)=J_i(x_\mu)} . \quad (1)$$

This correspondence has been formulated precisely in the case of a $\mathcal{N} = 4$ Super Yang-Mills theory on one side and type IIB string theory in the $AdS_5 \times S^5$. The coupling g_{YM} and the number of colors N_c of the field theory are related to the radius of the sphere S^5 and AdS_5 , string tension α'^{-1} and coupling g_{string} as well as to the 't Hooft constant λ' :

$$\frac{R^4}{\alpha'^2} = 4\pi g_{string} N_c = g_{YM}^2 N_c = \lambda' . \quad (2)$$

When we move from the $\mathcal{N} = 4$ Super Yang-Mills theory towards the real-life QCD decreasing the number of supersymmetries and destroying the initial conformal symmetry we start facing more and more complex theories on the gravity side of the correspondence. In order to explore QCD from the holographic point of view two basically different approaches have been devised, so-called "top-down" and "bottom-up". In the former, one starts with a string theory setup and works all the way down to warp the ten-dimensional geometry so that it reflects the dynamics of QCD on its four-dimensional boundary, e.g. [2], [3]. In the latter one formulates a minimal model that holographically reproduces the symmetries and dynamics of QCD. Such models were proposed in [4], [5], [6]. One of their advantages is that these simple five-dimensional toy-models allow us to study the basic properties of QCD without the complexity of the "top-down" approach. For instance, masses, decay rates and couplings of the

lightest mesons as well as the Gell-Mann–Oakes–Renner relationship for the pion mass have been studied in [4], the linear confinement – in [5] and the chiral symmetry breaking – in [7].

However simple those models may be, they need to reproduce a set of exact equations that describe the low-energy dynamics of QCD – the so-called low-energy theorems. These theorems are derived from axial Ward identities in QCD and are formulated for two-, three- and four-point correlators. Their complete list can be found in [8]. We will focus on the following theorems in the $N_f = 2$ case [9], [10]:

$$\begin{aligned} \frac{i}{V} \int d^4x d^4y \langle \delta_{ij} S_0(x) S_0(y) - P_i(x) P_j(y) \rangle &= -\frac{G_\pi^2 \delta_{ij}}{m_\pi^2} + \delta_{ij} \frac{B^2}{8\pi^2} (L_3 - 4L_4 + 3) \\ &= 2\delta_{ij} \int d\lambda \left(\frac{m \frac{\partial}{\partial m} \rho(\lambda, m)}{(\lambda^2 + m^2)} - \frac{2m^2 \rho(\lambda, m)}{(\lambda^2 + m^2)^2} \right), \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{i}{V} \int d^4x d^4y \langle S_i(x) S_j(y) - \delta_{ij} P_0(x) P_0(y) \rangle &= -8\delta_{ij} B^2 L_7 = \delta_{ij} \int d\lambda \frac{4m^2 \rho(\lambda, m)}{(\lambda^2 + m^2)^2} \\ &\quad - 2\delta_{ij} \frac{\int d^4x d^4y \langle Q(x) Q(y) \rangle}{m^2 V}, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{i}{V} \int d^4x d^4y \langle P_3(x) P_0(y) \rangle &= -\frac{G_\pi \tilde{G}_\pi}{m_\pi^2} = 2(m_u - m_d) m \int d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m^2)^2} \\ &\quad - (m_u - m_d) \frac{\int d^4x d^4y \langle Q(x) Q(y) \rangle}{m^3 V}. \end{aligned} \quad (5)$$

Here i, j are the adjoint flavor indices of the scalar and pseudoscalar currents $S(x)$ and $P(x)$; m_π is the pion mass; $G_\pi = 2F_\pi B = \frac{F_\pi m_\pi^2}{m}$ is its pseudoscalar decay constant; $\tilde{G}_\pi = 4(m_d - m_u) \frac{B^2 L_7}{F_\pi}$; $Q(x) = \frac{g^2 \theta}{32\pi^2} \text{tr} F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x)$ is the topological charge density; m is the quark mass, or, in the case of different quark masses, $m = \frac{1}{2}(m_u + m_d)$; V is the Euclidean four-volume; $\rho(\lambda, m)$ is the spectral density of the Dirac operator, and L_i are the constants of the next-to-leading order effective Chiral Lagrangian (see, e.g., [8, 11]).

The aim of this paper is to demonstrate that these theorems are satisfied in the dual AdS/QCD models in the chiral limit, and therefore we will focus on the leading terms in the expansion in powers of quark mass m of the relevant expressions. We leave the case of a finite mass to future studies.

In the next section of the paper **II** we compute the necessary two-point correlators from the AdS/QCD point of view, in the section **III** we discuss the implications of the AdS/QCD models for the Chiral Lagrangian, and in the section **IV** we present a way to calculate the

aforementioned spectral density in the AdS/QCD framework. Finally, in the section V we discuss the compatibility of the low-energy theorems with the AdS/QCD models.

A. Action and fields

We will consider the following five-dimensional action of the AdS/QCD models [4], [5], [12]:

$$S_{5D} = \int d^5x \sqrt{g} e^{-\Phi} \text{tr} \left\{ \Lambda^2 \left(|DX|^2 + \frac{3}{R^2} |X|^2 + \frac{\kappa}{R^2} |X|^4 \right) - \frac{1}{4g_5^2} (F_L^2 + F_R^2) \right\}. \quad (6)$$

with a metric $ds^2 = \frac{R^2}{z^2} (-dz^2 + dx_\mu dx^\mu) \equiv g_{MN} dx^M dx^N \equiv \frac{R^2}{z^2} \eta_{MN} dx^M dx^N$.

Here $g \equiv \det(g_{MN})$, Φ is a dilaton whose profile depends on the choice of a particular model. In the so-called hard-wall model [4] $\Phi(z) \equiv 0$ and the bulk five-dimensional space has a boundary at $z = z_m$ where a uniform Neumann boundary condition is imposed. In the soft-wall models [5], [12] the bulk space stretches over the whole range $0 \leq z < \infty$ and the dilaton profile is set to be asymptotically parabolic:

$$\Phi(z) \sim \lambda z^2, z \rightarrow \infty \quad (7)$$

so that the linear Regge trajectory is reproduced.

We introduce two gauge fields L_μ^a and R_μ^a in the adjoint representation of the $SU_L(N_f)$ and $SU_R(N_f)$ gauge groups respectively with curvatures $F_L = dL - iL \wedge L$, $F_R = dR - iR \wedge R$, and a scalar field $X^{\alpha\beta}$ with mass $-3/R^2$ in the bifundamental representation of $SU_L(N_f) \times SU_R(N_f)$ that interacts with the gauge fields: $DX = dX - iLX + iXR$.

According to the AdS/CFT prescription (1) the fields on the AdS boundary act as sources of the QCD currents:

$$\begin{aligned} L_\mu^a(x, z=0) &= \text{source of } \bar{q}_L(x) \gamma_\mu t^a q_L(x), \\ R_\mu^a(x, z=0) &= \text{source of } \bar{q}_R(x) \gamma_\mu t^a q_R(x), \\ \lim_{z \rightarrow 0} \frac{2}{z} X^{\alpha\beta}(x, z) &= \text{source of } \bar{q}_L^\alpha(x) q_R^\beta(x). \end{aligned} \quad (8)$$

In the AdS action g_5 is the 5D coupling constant defined by means of a comparison of the vector two-point correlator with the QCD sum rules [4], [13]: $\frac{g_5^2}{R} = \frac{12\pi^2}{N_c}$; Λ is the normalization factor for the scalar field which is fixed by comparing the pseudoscalar two-point correlator in

the large momentum regime with the sum rules [13], [7]: $\Lambda^2 R^3 = \frac{N_c}{4\pi^2}$. Hard-wall models generally do not include the effective potential of the scalar field $V(X) = \frac{\kappa}{R^2}|X|^4$ although it is necessary in the soft-wall models [12].

II. PSEUDOSCALAR AND SCALAR TWO-POINT CORRELATORS

We shall first discuss the simpler case of pseudoscalar currents.

A. Pseudoscalar currents

As compared to the scalar five-dimensional field, the pseudoscalar field is less sensitive to the shape of the effective potential $V(X)$. For that reason we will use a simpler hard-wall model [4], [7] while calculating the $\langle PP \rangle$ correlator. We will need the following part of the 5D action:

$$S_{5D} = \int d^5x \sqrt{g} \Lambda^2 \text{tr} \left(|\partial X|^2 + \frac{3}{R^2} |X|^2 \right). \quad (9)$$

According to the AdS/QCD prescription (8), $\left. \frac{2}{z} X^{\alpha\beta}(z, x) \right|_{z \rightarrow 0} \leftrightarrow \bar{q}_L^\alpha(x) q_R^\beta(x)$. This means that the pseudoscalar current corresponds to $i\bar{q}^\alpha(x) \gamma_5 q^\beta(x) \leftrightarrow \left. \frac{2}{z} \left(\frac{X - X^\dagger}{2i} \right)^{\alpha\beta} (z, x) \right|_{z \rightarrow 0}$.

The pseudoscalar two-point correlator is the second variation of the quadratic action with respect to the pseudoscalar sources. (See the appendix **A** for the details.)

$$\begin{aligned} i \langle P_i(q) P_j(p) \rangle &= -i e^{-iS_{5D}[J_P]} \frac{\delta}{\delta J_P^i(q)} \frac{\delta}{\delta J_P^j(p)} e^{iS_{5D}[J_P]} \Big|_{J_P=0} = \frac{\delta}{\delta J_P^i(q)} \frac{\delta}{\delta J_P^j(p)} S_{5D}[J_P] \\ &= \Lambda^2 R^3 \delta_{ij} \mathcal{D}(Q) \delta(p+q) + \text{terms originating from the } X \leftrightarrow L_\mu^a, R_\mu^a \text{ mixing}, \end{aligned} \quad (10)$$

where $\mathcal{D}(Q)$ is the integral of the on-shell Lagrangian with unity sources on the AdS boundary over dz , see (A8):

$$\mathcal{D}(Q) = \frac{1}{4} \left[\frac{1}{\epsilon^2} + \frac{4\sigma}{m} + Q^2 \log(Q^2 \epsilon^2) \right]. \quad (11)$$

Here we have introduced the UV cutoff at $z = \epsilon$.

The parameter σ is related to the quark condensate (A14):

$$C = \Lambda^2 R^3 \left(\frac{m}{4\epsilon^2} + \sigma \right). \quad (12)$$

This allows us to express the pseudoscalar correlator in (10) through the condensate:

$$i \langle P_i(q) P_j(-q) \rangle = \delta_{ij} \left\{ \frac{C}{m} + \frac{\Lambda^2 R^3}{4} Q^2 \log(Q^2 \epsilon^2) \right\} + \text{terms from the } X \leftrightarrow L_\mu^a, R_\mu^a \text{ mixing.} \quad (13)$$

The $X \leftrightarrow L_\mu^a, R_\mu^a$ mixing is proportional to Q^2 , so that when $Q^2 = 0$

$$i \langle P_i(0) P_j(0) \rangle = \delta_{ij} \frac{C}{m}. \quad (14)$$

One can see that the expression has the pole $\frac{1}{m} \propto \frac{1}{m_\pi^2}$ due to the pion exchange.

Let us also consider a particular $N_f = 2, m_u \neq m_d$ case. The difference from the previous consideration will arise in the expression of the bulk-to-boundary propagator and of the function $\mathcal{D}(Q) = \frac{1}{4} \left[\frac{1}{\epsilon^2} + Q^2 \log(Q^2 \epsilon^2) \right] + \sigma \cdot M^{-1}$ (c.f. (11)), where the matrices $M = \text{diag}(m_u, m_d)$, $\sigma = \text{diag}(\sigma_u, \sigma_d) \approx \sigma \cdot \mathbb{I}$. The second variation of the action due to pseudoscalar sources gives:

$$\begin{aligned} i \langle P_i(q) P_j(-q) \rangle &= \delta_{ij} \frac{\Lambda^2 R^3}{4} Q^2 \log(Q^2 \epsilon^2) \delta(p+q) + 2 \text{tr}(t^i \Sigma t^j) \delta(p+q) \\ &+ \text{terms from the } X \leftrightarrow L_\mu^a, R_\mu^a \text{ mixing.} \end{aligned} \quad (15)$$

The generalization of the result (14) is the following:

$$i \langle P_i(0) P_j(0) \rangle = \delta_{ij} \frac{C}{m} - \frac{C \Delta m}{2m^2} (\delta_{i0} \delta_{3j} + \delta_{j0} \delta_{3i} - i \delta_{i1} \delta_{2j} - i \delta_{j1} \delta_{2i}), \quad (16)$$

where $\Delta m \equiv m_u - m_d$.

B. Scalar currents

The model used in the previous subsection is insufficient when dealing with the scalar currents due to their sensitivity to the effective potential $V(X)$. Thus, in order to calculate the scalar two-point correlator we will consider a soft-wall AdS/QCD model [12] in which $V(X)$ arises naturally. In this model the dynamical fields have to vanish at $z \rightarrow \infty$ and the solutions of the equations of motion are one-parametric unlike (A5). This implies that in the absence of the quartic term in the action the solution for the scalar field would be proportional to its source – the quark mass – and would vanish in the chiral limit. This would not allow us to

disentangle the spontaneous chiral symmetry breaking from the explicit one. Introduction of the effective potential restores the correct chiral limit and the constant κ generates the mass splitting between axial and vector mesons, $\kappa \approx 15$ being the best fit to the radial spectra of the axial mesons [12]. The parameter λ in (7), responsible for the slope of the Regge trajectory, is determined to be $\lambda \approx 0.183 \text{ GeV}^2$ [12].

We will use the action in the form

$$S_{5D} = \int d^5x \sqrt{g} e^{-\Phi} \Lambda^2 \text{tr} \left(|\partial X|^2 + \frac{3}{R^2} |X|^2 + \frac{\kappa}{R^2} |X|^4 \right). \quad (17)$$

In the case when $X(z)$ is real and proportional to the unity matrix, $X(z) = \frac{\mathcal{V}(z)}{2} \cdot \mathbb{I}$, the nonlinear equation of motion assumes the form:

$$\partial_z \left(\frac{e^{-\Phi(z)} \partial_z \mathcal{V}(z)}{z^3} \right) + \frac{e^{-\Phi(z)}}{z^5} \left(3\mathcal{V}(z) + \frac{\kappa}{2} \mathcal{V}^3(z) \right) = 0. \quad (18)$$

The AdS/QCD prescription (8) implies that

$$\mathcal{V}(z) \sim mz + \sigma z^3, z \rightarrow 0. \quad (19)$$

Eq. (18) may be rewritten as an equation for the dilaton [12]:

$$\partial_z \Phi(z) = \frac{z^3}{\partial_z \mathcal{V}(z)} \left(\partial_z \left(\frac{\partial_z \mathcal{V}(z)}{z^3} \right) + \frac{3}{z^5} \mathcal{V}(z) + \frac{\kappa}{2z^5} \mathcal{V}^3(z) \right), \quad (20)$$

from which it follows that asymptotically

$$\mathcal{V}(z) \sim \gamma z, \gamma = 2\sqrt{\frac{\lambda}{\kappa}}. \quad (21)$$

The freedom in the behavior of the dilaton for small values of z enables us to choose any form of the function $\mathcal{V}(z)$ provided that it satisfies the conditions (19,21). For instance,

$$\mathcal{V}(z) = z \left(m + (\gamma - m) \tanh \left(\frac{\sigma z^2}{\gamma - m} \right) \right) \quad (22)$$

is a suitable choice. The dilaton will have an asymptotically parabolic profile that switches from $\frac{1}{4}\kappa m^2 z^2, z \rightarrow 0$ to $\lambda z^2, z \rightarrow \infty$ [12].

The nonlinear dependence of the classical solution $\mathcal{V}(z)$ on the mass m can be interpreted as a nonlinear form of the bulk-to-boundary propagator in this model. More exactly, the latter is defined by the whole set of functions $\mathcal{V}(z)$, $\frac{\partial}{\partial m} \mathcal{V}(z)$, $\frac{\partial^2}{\partial m^2} \mathcal{V}(z)$ etc. in the case of a source that is uniform (i.e. its four-momentum equals zero) and scalar (i.e. proportional to the

identity matrix in the flavor space $\mathbb{I}_{N_f \times N_f}$). Instead of a variation due to a scalar source we will be differentiating the action with respect to m .

The 5D on shell action (17) equals:

$$S_{5D \text{ cl.}} = \int dz \frac{d^4 q}{(2\pi)^4} N_f \Lambda^2 R^3 \left(\frac{1}{4} \mathcal{V} \mathcal{O}_{5D} \mathcal{V} - \frac{1}{16} \tilde{\kappa} \mathcal{V}^4 \right) + \int \frac{d^4 q}{(2\pi)^4} N_f \Lambda^2 R^3 \frac{1}{4} \mathcal{V} \mathcal{O}_{\partial 5D} \mathcal{V} \equiv S_{vol} + S_{surf}, \quad (23)$$

where $\mathcal{O}_{5D} = \partial_z \left(\frac{e^{-\Phi(z)} \partial_z \cdot}{z^3} \right) + 3 \frac{e^{-\Phi(z)}}{z^5}$, $\mathcal{O}_{\partial 5D} = \frac{e^{-\Phi(z)} \partial_z}{z^3} \cdot \Big|_{z=\epsilon \rightarrow 0}$, $\tilde{\kappa} = \kappa \frac{e^{-\Phi(z)}}{z^5}$.

On shell $\mathcal{O}_{5D} \mathcal{V} - \frac{1}{2} \tilde{\kappa} \mathcal{V}^3 = 0$ (c.f. (18)) and

$$S_{vol} = \frac{1}{16} N_f \Lambda^2 R^3 \kappa \int dz \frac{e^{-\Phi(z)}}{z^5} \mathcal{V}^4(z). \quad (24)$$

The second derivative of the on-shell action (17) with respect to the mass yields:

$$i \langle S(0) S(0) \rangle = \frac{1}{2} \partial_m^2 \mathcal{V} \frac{\delta S_{surf}}{\delta X} + \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{surf}}{\delta X^2} \partial_m \mathcal{V} + \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{vol}}{\delta X^2} \partial_m \mathcal{V}. \quad (25)$$

The details of the calculation can be found in the appendix **B**, eq. (B2 – B6).

The result given in (B7) is the following:

$$i \langle S_i(0) S_j(0) \rangle = \delta_{ij} \left\{ \frac{3}{8\pi^2} \lambda N_c A_0 + \frac{3}{16\pi^2} m \sqrt{\lambda \kappa} N_c A_1 + \frac{N_c}{4\pi^2 \epsilon^2} + \frac{3}{32\pi^2} m^2 \kappa N_c \left(A_2 - \frac{2}{3} + \log \left(\frac{2\pi^2 C \epsilon^2}{N_c} \sqrt{\frac{\kappa}{\lambda}} \right) - \frac{N_c}{2\pi^2 C} \sqrt{\frac{\lambda^3}{\kappa}} A_0 \right) \right\}, \quad (26)$$

where numerically

$$A_0 = 0.377, \quad A_1 = 0.977, \quad A_2 = -1.487. \quad (27)$$

III. THE QUARTIC PION LAGRANGIAN

The 5D effective action of the AdS/QCD model can be interpreted not only as the generating functional for the correlators of the QCD currents but also as a low-energy action of mesons. Indeed, if the five-dimensional fields on the boundary are sources of the QCD currents and these currents in their turn are sources of the corresponding mesons, one can Kaluza–Klein decompose the fields in the bulk and obtain an effective action for the modes of the 4D boundary fields.

These modes are proportional to the wavefunctions of the mesons with corresponding quantum numbers. Integrating out all the dynamics along the z axis we will obtain an effective low-energy action for the mesons which may be rewritten as a sum of the lowest-order terms of the Chiral Lagrangian. Since both Lagrangians possess the same symmetries we shall assume that one obtained from the AdS/QCD point of view reproduces the QCD Chiral Lagrangian.

The NLO part of the Chiral Lagrangian that we are interested in is the following (see, e.g., [8]):

$$\begin{aligned}\mathcal{L}_{\chi \text{ NLO}} = & L_1 \text{Tr}^2 (\partial_\mu U^\dagger \partial^\mu U) + L_2 \text{Tr} (\partial_\mu U^\dagger \partial_\nu U) \text{Tr} (\partial^\mu U^\dagger \partial^\nu U) + L_3 \text{Tr} (\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U) \\ & + L_4 \text{Tr} (D_\mu U^\dagger D^\mu U) \text{Tr} (U^\dagger \chi + \chi^\dagger U).\end{aligned}\quad (28)$$

We will focus on the pion dynamics and will therefore need to consider the axial fields in the bulk.

Let us consider the gauge sector of the hard-wall AdS/QCD action (c.f. (6)):

$$S_{5D} = \int d^5x \sqrt{g} \frac{1}{g_5^2} \text{tr} \left(-\frac{1}{4} (F_L^2 + F_R^2) \right), \quad (29)$$

where $F_L = dL - iL \wedge L$, $F_R = dR - iR \wedge R$, $DX = dX - iLX + iXR$, $X(z) = \frac{1}{2}v(z) \cdot 1_{N_f \times N_f}$, $v(z) \equiv mz + \sigma z^3$. We can redefine the action in terms of the vector and axial fields $L = V + A$, $R = V - A$ and chose a gauge $V_z = A_z = \partial^\mu V_\mu = 0$. The A_μ field is divided into transverse and longitudinal parts: $A_\mu = A_{\perp\mu} + \partial_\mu \phi$.

Since for the axial and pseudoscalar currents $\partial_\mu \mathcal{A}^\mu = 2mP$, then $-2m\phi(z=0, x)$ is the source of the pseudoscalar current. In its turn, the pseudoscalar current being the source of pions, $\langle 0 | P^a(p) | \pi^b(q) \rangle = iG_\pi \delta^{ab} \delta(p-q) = i \frac{C}{f_\pi} \delta^{ab} \delta(p-q)$, would suggest that $\phi^a(z=0, x)$ is proportional to $\pi^a(x)$. We will establish the exact proportionality later.

We can Kaluza-Klein decompose the fields V and ϕ :

$$V_\mu^a(z, x) = \sum_n V_\mu^{a(n)}(x) f_V^{(n)}(z), \quad \phi^a(z, x) = \sum_n \phi^{a(n)}(x) f_\phi^{(n)}(z). \quad (30)$$

The normalization is the following:

$$\int \frac{dz}{z} f^{(n)}(z) f^{(m)}(z) = \delta^{mn}. \quad (31)$$

The functions $V_\mu^{a(n)}(x)$ and $\phi^{a(n)}(x)$ are proportional to the four-dimensional wavefunctions of the vector and pseudoscalar mesons respectively. We shall focus on the lowest mode $\phi^{a(0)}(x)$ which corresponds to the pion field $\pi^a(x)$.

The functions $f(z)$ in their turn are the solutions of the equations of motion with the boundary conditions $f(0) = \partial_z f(z_m) = 0$. The lowest modes $f_\phi(z)$, $f_V(z)$ correspond to the π and ρ mesons. $f_V(z)$ has the following form:

$$f_V(z) = N_V z I_1(m_\rho z), \quad N_V^{-2} = \frac{z_m^2}{2} (I_1^2(m_\rho z_m) - I_2(m_\rho z_m) I_0(m_\rho z_m)). \quad (32)$$

After we integrate out all the dynamics along the z axis and leave only the lowest mode of the field $\phi(x)$ the five-dimensional action (29) yields:

$$\begin{aligned} S_{5D} \rightarrow \int d^4x \operatorname{tr} \left(N_\pi^2 \cdot \frac{1}{2} \partial_\mu \phi^{(0)}(x) \partial^\mu \phi^{(0)}(x) + g_{\phi^4} \cdot [\partial_\mu \phi^{(0)}(x), \partial_\nu \phi^{(0)}(x)] [\partial^\mu \phi^{(0)}(x), \partial^\nu \phi^{(0)}(x)] \right. \\ \left. + g_{m \phi^2} \cdot m \partial_\mu \phi^{(0)}(x) \partial^\mu \phi^{(0)}(x) \right). \end{aligned} \quad (33)$$

The quantities N_π , g_{ϕ^4} , $g_{m \phi^2}$ are the corresponding integrals over z and their explicit expressions can be found in the appendices **D** eq. (D2), **E** eq. (E4) and **D** eq. (D6) respectively.

The canonical normalization of the pion field in (33) demands that the proportionality between the pion and the ϕ assume the form:

$$N_\pi \phi^{a(0)}(x) = F_\pi \pi^a(x). \quad (34)$$

There are four terms in the next-to-leading order Chiral Lagrangian (28) that are relevant to us:

$$\begin{aligned} \mathcal{P}_1 &= \operatorname{Tr}^2 (\partial_\mu U^\dagger \partial^\mu U), \\ \mathcal{P}_2 &= \operatorname{Tr} (\partial_\mu U^\dagger \partial_\nu U) \operatorname{Tr} (\partial^\mu U^\dagger \partial^\nu U), \\ \mathcal{P}_3 &= \operatorname{Tr} (\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U), \\ \mathcal{P}_4 &= \operatorname{Tr} (D_\mu U^\dagger D^\mu U) \operatorname{Tr} (U^\dagger \chi + \chi^\dagger U) = 8 \frac{\Sigma}{F_\pi^2} m \partial_\mu \pi^a(x) \partial^\mu \pi^a(x) + \mathcal{O}(\pi^4). \end{aligned} \quad (35)$$

Analogously to [15] the Skyrme-like quartic term in (33) induced by the AdS dynamics is the lowest power in π of a linear combination of \mathcal{P}_i :

$$\begin{aligned} g_{\phi^4} \frac{F_\pi^4}{N_\pi^4} [\partial_\mu \pi(x), \partial_\nu \pi(x)] [\partial^\mu \pi(x), \partial^\nu \pi(x)] &= \sum_{i=1}^3 L_i \mathcal{P}_i + o(\pi^4, p^4), \\ \text{where } L_1 &= -\frac{g_{\phi^4}}{8} \frac{F_\pi^4}{N_\pi^4}, \quad L_2 = -\frac{g_{\phi^4}}{4} \frac{F_\pi^4}{N_\pi^4} = 2L_1, \quad L_3 = \frac{3g_{\phi^4}}{4} \frac{F_\pi^4}{N_\pi^4} = -6L_1. \end{aligned} \quad (36)$$

The relation between L_1, L_2, L_3 is in agreement with [14, 15], but, unlike in [14], we did not need to consider the $\rho\pi\pi$ interaction.

From (33) and (35) we obtain (see (D7)):

$$L_4 = \frac{F_\pi^4}{8\Sigma} g_m \phi^2 N_\pi^{-2}. \quad (37)$$

One can explicitly demonstrate that L_i are regular in the chiral limit and are $\propto N_c$.

IV. SPECTRAL DENSITY OF THE DIRAC OPERATOR

While both the correlation functions and the Chiral Lagrangian parameters may be obtained from the effective 5D action more or less straightforwardly, the spectral density of the Dirac operator is a more complex entity because of its fermionic nature. In order to be able to calculate it in dual holographic theories we have to express it through the partition function.

Let us consider a Yang-Mills theory of gluons A_μ with N_c colors and N_f flavors of quarks Ψ in the fundamental representation of $SU(N_c)$ with mass m and coupling g . We define the Dirac operator \hat{D} as follows:

$$\hat{D} \equiv \gamma^\mu (\partial_\mu + ig A_\mu). \quad (38)$$

Let $\{\lambda_n\}$ be the eigenvalues of the Dirac operator:

$$\text{for } \forall \lambda \in \{\lambda_n\} \exists \Psi : i\hat{D}\Psi = \lambda\Psi. \quad (39)$$

The spectrum is N_f times degenerate due to the global $SU(N_f)_V$ symmetry. \hat{D} and γ_5 anticommute, so that for every eigenvalue λ_n and eigenvector Ψ_n there is eigenvalue $-\lambda_n$ corresponding to $\gamma_5\Psi_n$.

We can define the Euclidean four-volume V and the corresponding spectral density of the Dirac operator:

$$\rho(\lambda) \equiv \frac{1}{V} \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A. \quad (40)$$

We shall introduce a regulatory mass μ and rewrite an infinite sum in terms of functional trace. The density was originally expressed in terms of a resolvent in [17].

$$\begin{aligned} \rho(\lambda) &= \frac{1}{V} \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A = \frac{1}{\pi V} \left\langle \lim_{\mu \rightarrow 0} \sum_n \frac{\mu}{\mu^2 + (\lambda - \lambda_n)^2} \right\rangle_A \\ &= \frac{-i}{2\pi V} \lim_{\mu \rightarrow 0} \left\langle \text{Tr}[i\hat{D} - \lambda - i\mu]^{-1} - \text{Tr}[i\hat{D} - \lambda + i\mu]^{-1} \right\rangle_A \\ &= \frac{1}{2\pi V} \lim_{\mu \rightarrow 0} \frac{\partial}{\partial \mu} \left\langle \log \text{Det}[i\hat{D} - \lambda - i\mu] + \log \text{Det}[i\hat{D} - \lambda + i\mu] \right\rangle_A \end{aligned} \quad (41)$$

Using a procedure called the "replica trick" ($\log z = \frac{\partial}{\partial n} z^n \big|_{n=0}$) we will be able to rewrite the logarithm of a determinant in terms of a partition function with the introduction of ghost quarks χ . Similar calculations can be seen in [16].

$$\begin{aligned}
\rho(\lambda) &= \frac{1}{\pi V} \lim_{\mu \rightarrow 0} \frac{\partial}{\partial \mu} \Re \left\langle \log \int \prod_{I=1}^{N_f} D\chi_I D\bar{\chi}_I \exp \left\{ - \int d^4x \bar{\chi}_I (i\hat{D} - \lambda - i\mu) \chi_I \right\} \right\rangle_A \quad (N_f \text{ } \chi \text{ flavors}) \\
&= \frac{1}{\pi V} \lim_{\mu \rightarrow 0} \frac{\partial}{\partial \mu} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \Re \left\langle \int \prod_{I=1}^{n \cdot N_f} D\chi_I D\bar{\chi}_I \exp \left\{ - \int d^4x \bar{\chi}_I (i\hat{D} - \lambda - i\mu) \chi_I \right\} \right\rangle_A \\
&= \frac{1}{\mathcal{Z}_{QCD} \pi V} \lim_{\mu \rightarrow 0} \frac{\partial}{\partial \mu} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \Re \int DA \prod_{J=1}^{N_f} D\Psi_J D\bar{\Psi}_J \prod_{I=1}^{n \cdot N_f} D\chi_I D\bar{\chi}_I \\
&\times \exp \left\{ i \int d^4x \bar{\chi}_I (i\hat{D} - \lambda - i\mu) \chi_I + \bar{\Psi}_J (i\hat{D} - m) \Psi_J - \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{g^2 \theta}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\}. \quad (42)
\end{aligned}$$

For further simplicity we will introduce the following effective actions:

$$\begin{aligned}
S_{QCD} &\equiv -i \log \mathcal{Z}_{QCD} = -i \log \int DA \prod_{J=1}^{N_f} D\Psi_J D\bar{\Psi}_J \\
&\times \exp \left\{ i \int d^4x \bar{\Psi}_J (i\hat{D} - m) \Psi_J - \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{g^2 \theta}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\}, \quad (43)
\end{aligned}$$

$$\begin{aligned}
S_{QCD+ghosts} &\equiv -i \log \mathcal{Z}_{QCD+ghosts} = -i \log \int DA \prod_{J=1}^{N_f} D\Psi_J D\bar{\Psi}_J \prod_{I=1}^{n \cdot N_f} D\chi_I D\bar{\chi}_I \\
&\times \exp \left\{ i \int d^4x \bar{\chi}_I (i\hat{D} - \mu - i\lambda) \chi_I + \bar{\Psi}_J (i\hat{D} - m) \Psi_J - \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{g^2 \theta}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \quad (44)
\end{aligned}$$

The expression (42) may now be simplified:

$$\begin{aligned}
\rho(\lambda, m) &= \frac{1}{\pi V} \Re e^{-iS_{QCD}} \left(-\frac{\partial}{\partial \mu} S \frac{\partial}{\partial n} S + i \frac{\partial}{\partial \mu} \frac{\partial}{\partial n} S \right) e^{iS_{QCD+ghosts}} \Big|_{\mu=n=0} \\
&= \frac{1}{\pi V} \Re \left(-\frac{\partial}{\partial \mu} \cdot \frac{\partial}{\partial n} \cdot + i \frac{\partial}{\partial \mu} \frac{\partial}{\partial n} \cdot \right) S_{QCD+ghosts}(\lambda, \mu, n, m) \Big|_{\mu=n=0}. \quad (45)
\end{aligned}$$

A. Spectral density in hard wall AdS/QCD

Let us denote the physical quarks Ψ_J , $J = 1 \dots N_f$, and the ghost quarks χ_I , $I = 1 \dots n \cdot N_f$ (See formula 42). We use the AdS/QCD prescription (8) to put a five-dimensional scalar field

$X^{\alpha\beta}(z, x)$, where $\alpha, \beta = 1 \dots (n+1)N_f$, into correspondence with the scalar currents:

$$\left. \frac{2}{z} X^{\alpha\beta}(z, x) \right|_{z \rightarrow 0} \leftrightarrow \bar{\Psi}_L^\alpha(x) \Psi_R^\beta(x), \quad \alpha, \beta = 1 \dots N_f, \quad (46)$$

$$\left. \frac{2}{z} X^{\alpha(\beta+N_f)}(z, x) \right|_{z \rightarrow 0} \leftrightarrow \bar{\Psi}_L^\alpha(x) \chi_R^\beta(x), \quad \alpha = 1 \dots N_f, \quad \beta = 1 \dots nN_f, \quad (47)$$

$$\left. \frac{2}{z} X^{(N_f+\alpha)\beta}(z, x) \right|_{z \rightarrow 0} \leftrightarrow \bar{\chi}_L^\alpha(x) \Psi_R^\beta(x), \quad \alpha = 1 \dots nN_f, \quad \beta = 1 \dots N_f, \quad (48)$$

$$\left. \frac{2}{z} X^{(N_f+\alpha)(N_f+\beta)}(z, x) \right|_{z \rightarrow 0} \leftrightarrow \bar{\chi}_L^\alpha(x) \chi_R^\beta(x), \quad \alpha = 1 \dots nN_f, \quad \beta = 1 \dots nN_f. \quad (49)$$

The Lagrangian (42) suggests that the Lagrangian of the five-dimensional model possesses a $SU(N_f) \times SU(nN_f)$ gauge symmetry. The action of the scalar field X is (6):

$$S_{5D} = \int d^5x \sqrt{g} \Lambda^2 \text{tr} \left(|\partial X|^2 + \frac{3}{R^2} |X|^2 \right). \quad (50)$$

The source for X is:

$$\frac{2}{\epsilon} X^{\alpha\beta}(\epsilon, x) = \text{diag}(\underbrace{m \dots m}_{N_f}, \underbrace{\lambda + i\mu \dots \lambda + i\mu}_{nN_f}) \equiv m \cdot \mathbb{P}^\Psi + (\lambda + i\mu) \cdot \mathbb{P}^\chi,$$

where $\mathbb{P}^\Psi \equiv \delta_{\alpha\beta}$ iff $\alpha, \beta = 1 \dots N_f$ is a projector on the quark states Ψ^α in flavor space, $\mathbb{P}^\chi \equiv \delta_{\alpha\beta}$ iff $\alpha, \beta = N_f + 1 \dots (n+1)N_f$ is an analogous projector on the ghost states χ^β .

The function $X(z, x)$ in the bulk is expressed in terms of the source via the bulk-to-boundary propagator, similar to (A5):

$$\begin{aligned} X_{\alpha\beta}(z, x) &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \mathcal{K}_{\alpha\gamma}(z, q) \frac{2}{\epsilon} X_{\gamma\beta}(\epsilon, q) \\ &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \mathcal{K}_{\alpha\gamma}(z, q) \delta(q) [m \mathbb{P}_{\gamma\beta}^\Psi + (\lambda + i\mu) \mathbb{P}_{\gamma\beta}^\chi] \end{aligned} \quad (51)$$

(c.f. (A7)).

We assume that the bulk-to-boundary has the following form:

$$\mathcal{K}_{\alpha\gamma}(z, q) = \mathcal{K}(z, q) \mathbb{P}_{\alpha\gamma}^\Psi + \tilde{\mathcal{K}}(z, q) \mathbb{P}_{\alpha\gamma}^\chi. \quad (52)$$

In QCD with quarks Ψ with mass m and condensate $\Sigma = C N_f = \sigma N_f \Lambda^2 R^3$ the field X in the bulk equals $X(z) = \frac{1}{2} (mz + \sigma z^3) \equiv \frac{1}{2} v(z)$. A QCD-like theory with ghosts χ with mass $\lambda + i\mu$ should have the field X of the same form: $X(z) = \frac{1}{2} ((\lambda + i\mu)z + \tilde{\sigma} z^3)$, where $\tilde{\sigma} = \frac{\tilde{C}}{\Lambda^2 R^3}$,

\tilde{C} being the condensate of ghosts. In the first approximation $\tilde{C} = C$, but this equality is imprecise because of the mass-dependence of the condensate. This leads us to a conclusion that the quantity $\tilde{\mathcal{D}}(Q)$ (analogous to $\mathcal{D}(Q)$ from subsection II eq. (11) for quarks Ψ where that quantity was defined as the integral over dz of the on-shell Lagrangian with unity sources on the AdS boundary) equals:

$$\tilde{\mathcal{D}}(Q) = \frac{\tilde{C}}{\Lambda^2 R^3 (\lambda + i\mu)} + \frac{Q^2}{4} \log(Q^2 \epsilon^2). \quad (53)$$

From Lagrangian in (42) it follows that the current $\bar{\chi}\chi$ has to have a complex-valued source, while $\bar{\chi}\gamma_5\chi$ has to have none. This implies that $X(\epsilon, x)$ has to be complex and $X(\epsilon, x) - X^\dagger(\epsilon, x) = 0$. The only way to do this is to temporarily introduce another field Y instead of X^\dagger and to treat X and Y independently, so that in general $Y^\dagger \neq X$, in fact, $Y = X$.

The action may be expressed in terms of the sources:

$$\begin{aligned} S_{5D} &= \Lambda^2 R^3 \text{tr} \int \frac{d^4 q}{(2\pi)^4} \left[\mathcal{D}(Q) \mathbb{P}^\Psi + \tilde{\mathcal{D}}(Q) \mathbb{P}^\chi \right] \\ &\times \left[m \mathbb{P}^\Psi + (\lambda + i\mu) \mathbb{P}^\chi \right] \delta(q) \times \left[m \mathbb{P}^\Psi + (\lambda + i\mu) \mathbb{P}^\chi \right] \delta(q) \\ &= \Lambda^2 R^3 \delta(0) \left(m^2 N_f \mathcal{D}(0) + (\lambda + i\mu)^2 n N_f \tilde{\mathcal{D}}(0) \right) = N_f V \left(mC + (\lambda + i\mu)n\tilde{C} \right). \end{aligned} \quad (54)$$

To calculate the spectral density we will use the formula (45). Since S_{5D} is a linear function of $\mu \cdot n$, the first term $\frac{\partial}{\partial \mu} S \frac{\partial}{\partial n} S \Big|_{n=0} = 0$. Hence,

$$\rho(\lambda) = \frac{1}{\pi V} \Re i \frac{\partial}{\partial \mu} \frac{\partial}{\partial n} S_{5D} \Big|_{\mu=n=0} = \frac{1}{\pi V} \Re N_f V \left[-\tilde{C} + i(\lambda + i\mu) \frac{\partial}{\partial \mu} \tilde{C} \right] \Big|_{\mu=0}. \quad (55)$$

Let us denote $C(m) = \sum_{n=0}^{\infty} C_n m^n$ for $\Re(m) > 0$. The expression (55) yields:

$$\rho(\lambda) = -\frac{N_f}{\pi} \sum_{n=0}^{\infty} C_n (n+1) \lambda^n = -\frac{1}{\pi} \Sigma_0 - \frac{1}{\pi} \sum_{n=1}^{\infty} \Sigma_n (n+1) \lambda^n \quad (\Re(\lambda) > 0), \quad (56)$$

where $\Sigma(m) = \sum_{n=0}^{\infty} \Sigma_n m^n$ for $\Re(m) > 0$, $\Sigma_0 \equiv \Sigma|_{m=0}$.

Making use of the formula from [13] that describes the mass-dependence of the quark condensate $\Sigma(m)$

$$\Sigma(m) = \Sigma_0 \left(1 - \frac{3m_\pi^2 \log m_\pi^2 / \mu_{had}^2}{32\pi^2 F_\pi^2} + \dots \right), \quad (57)$$

we obtain:

$$\rho(\lambda) = -\frac{1}{\pi} \Sigma_0 \left(1 - \frac{3\Sigma_0}{8N_f \pi^2 F_\pi^4} |\lambda| - \frac{3\Sigma_0}{4N_f \pi^2 F_\pi^4} |\lambda| \log |\lambda / \tilde{\mu}_{had}| \right). \quad (58)$$

As one may notice, firstly, the result satisfies the Casher–Banks identity [18]. Secondly, it reproduces up to the factor $\propto N_f^2 - 4$ a well-known formula from QCD, first derived in [19],

$$\frac{\rho'(0)}{\rho(0)} \propto \frac{\Sigma}{F_\pi^4} \quad (59)$$

for the term linear in λ .

Nevertheless, eq. (58) does not describe the dependence of the spectral density on the mass m and does not include the terms λ^2 and higher powers of λ .

V. LOW-ENERGY THEOREMS FROM THE HOLOGRAPHIC POINT OF VIEW

Having obtained in sections II – IV the expressions for all the necessary ingredients of the low-energy theorems in the AdS/QCD framework we shall determine now whether (and how accurately) these theorems are satisfied in the dual models.

We will be using the following result for the two-point correlator of the topological charge density [21]:

$$\frac{\int d^4x d^4y \langle Q(x)Q(y) \rangle}{V} = \frac{mBF_\pi^2}{2} + o(m) = mC + o(m), \quad (60)$$

that was reproduced in AdS/QCD in [20].

Let us consider the first low-energy theorem (3). One can see that all its three sides have a pole at $m \rightarrow 0$, and the pole residue in the left-hand side C (14) equals that of the central side. Indeed,

$$\frac{G_\pi^2}{m_\pi^2} = \frac{4C^2}{m_\pi^2 F_\pi^2} = \frac{C}{m}. \quad (61)$$

As for the right-hand side, from (56) we obtain

$$\int d\lambda \frac{4m^2 \rho(\lambda, m)}{(\lambda^2 + m^2)^2} = \frac{4Cm^2}{\pi} \int d\lambda \frac{1}{(\lambda^2 + m^2)^2} + \mathcal{O}(1) = \frac{C}{m} + \mathcal{O}(1), \quad m \rightarrow 0, \quad (62)$$

which agrees with the residues of the left-hand and central sides. As for the finite part of the left-hand and central sides, one can straightforwardly check that both $\langle S_i(0)S_j(0) \rangle$ (26) and $B^2(L_3 - 4L_4 + 3)$ (36, 37) are $\propto N_c$.

In the case of the second low-energy theorem (4) one can also observe the similar analytical structure of the left- and right-hand sides at the $m \rightarrow 0$ pole, the residues are again in agreement (see (61, 60))

As for the third low-energy theorem (5), one can see that the left-hand side defined by (16) (60), coincides with the right-hand side when $m \rightarrow 0$:

$$\frac{2Cm\Delta m}{\pi} \int d\lambda \frac{1}{(\lambda^2 + m^2)^2} - \frac{\Delta m}{m^3} Cm + \mathcal{O}(1) = -\frac{C\Delta m}{2m^2} + \mathcal{O}(1). \quad (63)$$

VI. DISCUSSION

As we can see, simple hard-wall or soft-wall AdS/QCD models allow us to test their compatibility with the low-energy-theorems of QCD. Indeed, using the dual description of the quantum field theory in the strong coupling regime we have demonstrated that the theorems are satisfied in the chiral limit when the quark and pion masses tend to zero. The zero-momentum two-point correlators of the pseudoscalar currents precisely coincide with the corollary of the theorems (61, 62, 63), and moreover the result (14) for the correlation function $\langle P_i(0)P_j(0) \rangle$ independently agrees with the field theory result (61).

Holographic models also induce low-energy dynamics of four-dimensional fields on the AdS boundary that can be put into correspondence with meson wavefunctions with suitable quantum numbers. This allows us to calculate the parameters of the effective low-energy Chiral Lagrangian in those models (36, 37). The established relation between the coefficients L_1, L_2, L_3 (36) reproduces the result [15] in the "top-down" Sakai–Sugimoto model [2].

Finally, we succeeded in expressing the spectral density of the Dirac operator in terms of a partition function of a theory which includes QCD and ghost quark fields (42, 45). This allowed us to calculate the density in the AdS/QCD framework, the result (58) agreeing with the Casher–Banks identity [18] and reproducing the general properties of the QCD formula [19] in the linear order in λ .

Being for the most part simplified, AdS/QCD models such as [4], [12] and [5] need further improvement and sharpening. One of their major shortcomings is the significant dependence of the results on the particular choice of the background geometry, presence of the dilaton and its profile, form of the effective potentials etc. For instance, the two-point correlator of scalar currents calculated in the hard-wall model [4], significantly differs from the expression (26) and has an unphysical pole structure. On the other hand, the soft-wall model [12] has not been yet directly generalized to describe quark flavors with different masses, and therefore in its present form does not allow us to calculate the spectral density using the formula (45). We intend to study the dependence of the AdS/QCD results for the coefficients of the NLO

Chiral Lagrangian on the choice of the model (hard-wall/ soft-wall) elsewhere, although at the moment it seems that the hard-wall model does not reproduce the coefficient L_7 .

All the aforementioned suggests that studying the AdS/QCD models by means of testing them with exact equations established on the field theory side of the correspondence is quite a promising task and will hopefully lead to the refinement of those models. Alongside with the advancement of the "top-down" approach that produces more rigorous results at a cost of significant complexity, this should reveal us the possible structure of the ultimate dual of the QCD.

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APPENDIX A: THE VARIATION OF THE ON-SHELL 5D ACTION WITH RESPECT TO PSEUDOSCALAR SOURCES

We consider the scalar sector of the 5D action (6):

$$S_{5D} = \int d^5x \sqrt{g} \Lambda^2 \text{tr} \left(|\partial X|^2 + \frac{3}{R^2} |X|^2 \right), \quad (\text{A1})$$

see (9).

If X is proportional to the unity $\mathbb{I}_{N_f \times N_f}$ matrix, the action in components is rewritten as follows:

$$S_{5D} = \int d^5x N_f \Lambda^2 R^3 \left[\frac{\partial_\mu X \partial^\mu X - (\partial_z X)^2}{z^3} + \frac{3}{z^5} X^2 \right]. \quad (\text{A2})$$

The equation of motion for the bulk-to-boundary propagator $\mathcal{K}(z, x - y)$ is:

$$\partial_z \left(\frac{1}{z^3} \partial_z \mathcal{K}(z, x - y) \right) - \frac{1}{z^3} \square \mathcal{K}(z, x - y) + \frac{3}{z^5} \mathcal{K}(z, x - y) = 0, \quad \lim_{z \rightarrow 0} \frac{2}{z} \mathcal{K}(z, x - y) = \delta(x - y.) \quad (\text{A3})$$

The solution in the 4D Euclidean momentum space is

$$\mathcal{K}(z, Q) = Q^2 z^2 (AK_1(Qz) + BI_1(Qz)). \quad (\text{A4})$$

The boundary condition implies that $A = \frac{1}{2Q}$, and the $Q \rightarrow 0$ limit – that $B = \frac{\sigma}{mQ^3}(1 + \mathcal{O}(1))$, where σ is proportional to the chiral condensate $C, \langle \bar{q}^\alpha q^\beta \rangle \equiv \Sigma^{\alpha\beta} \equiv C\delta^{\alpha\beta}$. Thus

$$\mathcal{K}(z, Q) = \frac{Qz^2}{2}K_1(Qz) + \frac{\sigma z^2}{mQ}I_1(Qz). \quad (\text{A5})$$

The source of the pseudoscalar current $J_P^{\alpha\beta}(x), \alpha, \beta = 1 \dots N_f$, is related to the boundary value of the X field on the regulatory UV brane:

$$\frac{2}{\epsilon}X^{\alpha\beta}(\epsilon, x) = iJ_P^{\alpha\beta}(x) + m\delta^{\alpha\beta}, \epsilon \rightarrow 0, \quad (\text{A6})$$

allowing us to express X in terms of J :

$$\begin{aligned} X_{\alpha\beta}(z, x) &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \mathcal{K}(z, q) \left[iJ_P^i(q) + \sqrt{2N_f}m\delta(q)\delta^{0i} \right] t_{\alpha\beta}^i, \\ X_{\alpha\beta}^\dagger(z, x) &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \mathcal{K}(z, -q) \left[-iJ_P^i(-q) + \sqrt{2N_f}m\delta(q)\delta^{0i} \right] t_{\alpha\beta}^i, \end{aligned} \quad (\text{A7})$$

where $t_{\alpha\beta}^i$ are the basis Hermitian $N_f \times N_f$ matrices, $i = 0 \dots N_f^2 - 1$.

If we denote the integral over z as:

$$\begin{aligned} \mathcal{D}(Q) &= \int dz \left\{ -\frac{\partial_z \mathcal{K}(z, Q)^2}{z^3} + \left(-\frac{Q^2}{z^3} + \frac{3}{z^5} \right) \mathcal{K}(z, Q)^2 \right\} = \frac{\mathcal{K}(z, Q) \partial_z \mathcal{K}(z, Q)}{z^3} \Big|_{z=\epsilon} \\ &= \frac{1}{4} \left[\frac{1}{\epsilon^2} + \frac{4\sigma}{m} + Q^2 \log(Q^2 \epsilon^2) \right], \end{aligned} \quad (\text{A8})$$

the action may be rewritten in terms of the pseudoscalar sources:

$$\begin{aligned} S_{5D}[J_P] &= \Lambda^2 R^3 \frac{\delta_{ab}}{2} \int \frac{d^4k}{(2\pi)^4} \mathcal{D}(k) \left[iJ_P^a(k) + \sqrt{2N_f}m\delta(k)\delta^{0a} \right] \left[-iJ_P^b(-k) + \sqrt{2N_f}m\delta(k)\delta^{0b} \right] \\ &\quad + S_\phi[J_P], \end{aligned} \quad (\text{A9})$$

where $S_\phi[J_P]$ is the action for the longitudinal component of the axial field ϕ which mixes with X [7].

The pseudoscalar two-point correlator is the second variation of the quadratic action:

$$\begin{aligned} i \langle P_i(q) P_j(p) \rangle &= -i e^{-iS_{5D}[J_P]} \frac{\delta}{\delta J_P^i(q)} \frac{\delta}{\delta J_P^j(p)} e^{iS_{5D}[J_P]} \Big|_{J_P=0} \\ &= \frac{\delta}{\delta J_P^i(q)} \frac{\delta}{\delta J_P^j(p)} S_{5D}[J_P] = \Lambda^2 R^3 \delta_{ij} \mathcal{D}(Q) \delta(p+q) + \text{terms originating from } S_\phi[J_P] \end{aligned} \quad (\text{A10})$$

In order to establish the precise connection between σ and the condensate we will calculate the latter as a variation of the 5D action due to a scalar source in the same way as we have calculated the pseudoscalar correlator [7]. In that case the boundary condition is the following:

$$\frac{2}{\epsilon} X^{\alpha\beta}(\epsilon, x) = J_S^{\alpha\beta}(x) + m\delta^{\alpha\beta}, \quad (\text{A11})$$

and the 5D action is

$$S_{5D}[J_S] = \Lambda^2 R^3 \frac{\delta_{ab}}{2} \int \frac{d^4 k}{(2\pi)^4} \mathcal{D}(k) \left[J_S^a(k) + \sqrt{2N_f} m \delta(k) \delta^{0a} \right] \left[J_S^b(-k) + \sqrt{2N_f} m \delta(k) \delta^{0b} \right] \quad (\text{A12})$$

Variation of the action yields:

$$\begin{aligned} \Sigma_{\alpha\beta}(q) &= \Sigma^a(q) t_{\alpha\beta}^a = -i t_{\alpha\beta}^a e^{-iS_{5D}[J_S]} \frac{\delta}{\delta J_S^a(q)} e^{iS_{5D}[J_S]} \Big|_{J_S=0} \\ &= t_{\alpha\beta}^a \frac{\delta}{\delta J_S^a(q)} S_{5D}[J_S] \Big|_{J_S=0} = t_{\alpha\beta}^a \Lambda^2 R^3 \frac{\delta_{ij}}{2} 2\mathcal{D}(k) \delta_{ai} \delta(k-q) \sqrt{2N_f} m \delta(k) \delta_{0j} \\ &= t_{\alpha\beta}^0 \sqrt{2N_f} \Lambda^2 R^3 m \mathcal{D}(0) \delta(q) = \delta_{\alpha\beta} \Lambda^2 R^3 \left(\frac{m}{4\epsilon^2} + \sigma \right) \delta(q). \end{aligned} \quad (\text{A13})$$

Thus,

$$\Sigma_{\alpha\beta} \equiv C \delta_{\alpha\beta} = \delta_{\alpha\beta} \Lambda^2 R^3 \left(\frac{m}{4\epsilon^2} + \sigma \right) \Rightarrow C = \Lambda^2 R^3 \left(\frac{m}{4\epsilon^2} + \sigma \right). \quad (\text{A14})$$

This allows us to express the pseudoscalar correlator through the condensate:

$$i \langle P_i(q) P_j(-q) \rangle = \delta_{ij} \left\{ \frac{C}{m} + \frac{\Lambda^2 R^3}{4} Q^2 \log(Q^2 \epsilon^2) \right\} + \text{terms originating from } S_\phi[J_P]. \quad (\text{A15})$$

APPENDIX B: THE VARIATION OF THE ON-SHELL 5D ACTION WITH RESPECT TO SCALAR SOURCES

In this appendix we will discuss second derivative of the on-shell action (17) with respect to the mass:

$$i \langle S(0) S(0) \rangle = \frac{1}{2} \partial_m^2 \mathcal{V} \frac{\delta S_{surf}}{\delta X} + \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{surf}}{\delta X^2} \partial_m \mathcal{V} + \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{vol}}{\delta X^2} \partial_m \mathcal{V}. \quad (\text{B1})$$

The first term here

$$\frac{1}{2} \partial_m^2 \mathcal{V} \frac{\delta S_{surf}}{\delta X} = \frac{1}{2} \left(\partial_m^2 \mathcal{V}(z) \frac{e^{-\Phi(z)} \partial_z}{z^3} \mathcal{V}(z) + \mathcal{V}(z) \frac{e^{-\Phi(z)} \partial_z}{z^3} \partial_m^2 \mathcal{V}(z) \right) \Big|_{z=\epsilon} = 0, \quad (\text{B2})$$

because $\partial_m^2 \mathcal{V}(z) = -\frac{2\sigma^2 z^5}{(\gamma - m)^3} \frac{\sinh\left(\frac{\sigma z^2}{\gamma - m}\right)}{\cosh^3\left(\frac{\sigma z^2}{\gamma - m}\right)} \sim z^7, z \rightarrow 0, \mathcal{V}(z) \sim z, z \rightarrow 0$.

The second term in (25)

$$\frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{surf}}{\delta X^2} \partial_m \mathcal{V} = \frac{N_f \Lambda^2 R^3}{4} \partial_m \mathcal{V}(z) \frac{e^{-\Phi(z)} \partial_z}{z^3} \partial_m \mathcal{V}(z) \Big|_{z=\epsilon} = N_f \Lambda^2 R^3 \left(\frac{1}{\epsilon^2} - \frac{1}{4} \kappa m^2 \right). \quad (\text{B3})$$

The third term in (25) is the most involved.

$$\begin{aligned} \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{vol}}{\delta X^2} \partial_m \mathcal{V} &= \frac{3}{4} N_f \Lambda^2 R^3 \kappa \int_{\epsilon}^{\infty} dz \frac{e^{-\Phi(z)}}{z^5} \mathcal{V}^2(z) \partial_m \mathcal{V}^2(z) \\ &= \frac{3}{8} N_f \Lambda^2 R^3 \kappa \gamma^2 \int_{\epsilon_x}^{\infty} dx \frac{e^{-\Phi(z(x))}}{x} \left[\tanh x + \frac{m}{\gamma} (1 - \tanh x) \right]^2 [1 - \tanh x + x \cosh^{-2} x]^2 \end{aligned} \quad (\text{B4})$$

where in the last integral we have introduced a dimensionless variable $x = \frac{\sigma z^2}{\gamma - m}, \epsilon_x = \frac{\sigma \epsilon^2}{\gamma - m}$.

The functions

$$\begin{aligned} f_0(x) &= \tanh^2 x \left(1 - \tanh x + \frac{x}{\cosh^2 x} \right)^2 / x, \\ f_1(x) &= 2 \tanh x (1 - \tanh x) \left(1 - \tanh x + \frac{x}{\cosh^2 x} \right)^2 / x, \\ f_2(x) &= (1 - \tanh x)^2 \left(1 - \tanh x + \frac{x}{\cosh^2 x} \right)^2 / x, \end{aligned}$$

decrease rapidly when $x \gtrsim 1$ and we will use the $z \rightarrow 0$ asymptotic behavior of the dilaton:

$$\Phi(z(x)) = \frac{\kappa m^2}{4} z^2 = \frac{\kappa m^2 (\gamma - m)}{4\sigma} x \equiv \varkappa x.$$

Numerically

$$A_0 = \int_0^{\infty} dx f_0(x) = 0.377, \quad A_1 = \int_0^{\infty} dx f_1(x) = 0.977, \quad A_2 = \int_0^{\infty} dx f_2(x) \Big|_{reg} = -1.487. \quad (\text{B5})$$

As a result,

$$\begin{aligned} \frac{1}{4} \partial_m \mathcal{V} \frac{\delta^2 S_{vol}}{\delta X^2} \partial_m \mathcal{V} &= \frac{3}{8\pi^2} \lambda N_f N_c A_0 + \frac{3}{16\pi^2} m \sqrt{\lambda \kappa} N_f N_c A_1 \\ &\quad + \frac{3}{32\pi^2} m^2 \kappa N_f N_c \left(A_2 + \log \left(\frac{2\pi^2 C \epsilon^2}{N_c} \sqrt{\frac{\kappa}{\lambda}} \right) - \frac{N_c}{2\pi^2 C} \sqrt{\frac{\lambda^3}{\kappa}} A_0 \right). \end{aligned} \quad (\text{B6})$$

We finally obtain from (25, B2, B3, B4, B6):

$$\begin{aligned} i \langle S_i(0) S_j(0) \rangle &= \delta_{ij} \left\{ \frac{3}{8\pi^2} \lambda N_c A_0 + \frac{3}{16\pi^2} m \sqrt{\lambda \kappa} N_c A_1 + \frac{N_c}{4\pi^2 \epsilon^2} \right. \\ &\quad \left. + \frac{3}{32\pi^2} m^2 \kappa N_c \left(A_2 - \frac{2}{3} + \log \left(\frac{2\pi^2 C \epsilon^2}{N_c} \sqrt{\frac{\kappa}{\lambda}} \right) - \frac{N_c}{2\pi^2 C} \sqrt{\frac{\lambda^3}{\kappa}} A_0 \right) \right\}. \end{aligned} \quad (\text{B7})$$

APPENDIX C: THE LOWEST KALUZA–KLEIN MODE OF THE FIELD ϕ

In this section we will explicitly derive the formula for the lowest Kaluza–Klein mode $f_\phi^{(0)} \equiv f_\phi$ of the field ϕ by solving the corresponding equations of motion perturbatively up to the first order in m and $m_\pi^2 \propto m$.

In order to find the first-order corrections to the chiral limit of the solution f_ϕ we will solve the following system of equations [4], [7] of motion perturbatively:

$$\partial_z \left(\frac{1}{z} \partial_z \phi \right) + k^2 \frac{v^2(z)}{z^3} (\pi - \phi) = 0 \quad (\text{C1})$$

$$m_\pi^2 \partial_z \phi + k^2 \frac{v^2(z)}{z^2} \partial_z \pi = 0. \quad (\text{C2})$$

Here $k^2 = \Lambda^2 R^2 g_5^2 = 3$, $\Lambda^2 R^3 = \frac{N_c}{4\pi^2}$, $\Lambda^2 R^3 \sigma = C$. The small parameter of the perturbative solution is m and $m_\pi^2 \propto m$. The boundary conditions are:

$$\pi(0) = 1, \quad \phi(0) = \partial_z \phi(z_m) = 0. \quad (\text{C3})$$

We cannot impose uniform zero boundary conditions on both functions ϕ and π without making the equations incompatible.

The solution in the chiral limit is:

$$\pi^{(0)}(z) = \text{const} = \pi(0) = 1, \quad (\text{C4})$$

$$\begin{aligned} \phi^{(0)}(z) = & 1 - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} z I_{1/3}(k\sigma z^3) \\ & - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} z K_{1/3}(k\sigma z^3). \end{aligned} \quad (\text{C5})$$

We will impose the boundary conditions (C3) at $z = 0$ and $z = z_m$ upon the Green function $\mathcal{G}(z, z')$.

The Green function is a solution of the following equation (C1):

$$\left(\partial_z \frac{1}{z} \partial_z - k^2 \sigma^2 z^3 \right) \mathcal{G}(z, z') = \delta(z - z'), \quad \mathcal{G}(0, z') = 0, \quad \partial_z \mathcal{G}(z, z')|_{z=z_m} = 0, \quad (\text{C6})$$

$$\begin{aligned}
\mathcal{G}(z, z') &= -\frac{zz'}{t'} \cdot \frac{1}{I_{-2/3}(t_m) \{K_{1/3}(t')I_{-2/3}(t') + K_{-2/3}(t')I_{1/3}(t')\}} \\
&\times \left\{ K_{1/3}(t')K_{-2/3}(t_m)I_{1/3}(t) + \theta(t-t')I_{1/3}(t')I_{-2/3}(t_m)K_{1/3}(t) \right. \\
&+ \left. \theta(t'-t)K_{1/3}(t')I_{-2/3}(t_m)I_{1/3}(t) \right\}, \quad t = k\sigma z^3, \quad t' = k\sigma z'^3, \quad t_m = k\sigma z_m^3. \quad (C7)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{zz'}{t'} \cdot \frac{1}{I_{-2/3}(t_m) \{K_{1/3}(t')I_{-2/3}(t') + K_{-2/3}(t')I_{1/3}(t')\}} \\
&\times \left\{ K_{1/3}(t') \left(K_{-2/3}(t_m) + I_{-2/3}(t_m) \right) I_{1/3}(t) \right. \\
&+ \left. \theta(t-t')I_{-2/3}(t_m) \left(I_{1/3}(t')K_{1/3}(t) - K_{1/3}(t')I_{1/3}(t) \right) \right\} \quad (C8)
\end{aligned}$$

$$\equiv \mathcal{F}_0(z') z I_{1/3}(k\sigma z^3) + \theta(z-z') \left(\mathcal{F}_I(z') z I_{1/3}(k\sigma z^3) + \mathcal{F}_K(z') z K_{1/3}(k\sigma z^3) \right). \quad (C9)$$

The first-order correction to $\pi(z)$ is (C2):

$$\pi^{(1)}(z) = -\frac{m_\pi^2}{k^2} \int_0^z du \frac{u^2 \partial_u \phi^{(0)}(u)}{v^2(u)}, \quad (C10)$$

and the first order correction to $\phi(z)$ is (C1):

$$\phi^{(1)}(z) = \int_0^{z_m} dw \mathcal{G}(z, w) \left(2m\sigma k^2 w (\phi^{(0)}(w) - \pi^{(0)}(w)) - k^2 \sigma^2 w^3 \pi^{(1)}(w) \right). \quad (C11)$$

We have retained the terms $\propto m, m^2$ in the $v^2(u)$ function in the expression (C10) for the sake of convergence of the integral although we exceed the necessary accuracy level.

Substituting (C5, C4, C10) into (C11) and using the form (C9) we obtain:

$$\begin{aligned}
\phi^{(1)}(z) &= -2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} \int_0^{z_m} dw \mathcal{G}(z, w) \left\{ 2m\sigma k^2 w^2 \left(I_{1/3}(k\sigma w^3) \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} \right. \right. \\
&+ \left. \left. K_{1/3}(k\sigma w^3) \right) + 3k\sigma^3 m_\pi^2 w^3 \int_0^w du \frac{u^5}{v^2(u)} \right. \\
&\times \left. \left(I_{-2/3}(k\sigma u^3) \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} - K_{-2/3}(k\sigma u^3) \right) \right\} \quad (C12)
\end{aligned}$$

$$= \mu_0 \cdot z I_{1/3}(k\sigma z^3) + \mu_I(z) \cdot z I_{1/3}(k\sigma z^3) + \mu_K(z) \cdot z K_{1/3}(k\sigma z^3). \quad (C13)$$

Here we have introduced the following notations (C8, C9):

$$\mu_0 = \int_0^{z_m} dw \mathcal{F}_0(w) \left(2m\sigma k^2 w (\phi^{(0)}(w) - \pi^{(0)}(w)) - k^2 \sigma^2 w^3 \pi^{(1)}(w) \right), \quad (\text{C14})$$

$$\mu_I(z) = \int_0^z dw \mathcal{F}_I(w) \left(2m\sigma k^2 w (\phi^{(0)}(w) - \pi^{(0)}(w)) - k^2 \sigma^2 w^3 \pi^{(1)}(w) \right), \quad (\text{C15})$$

$$\mu_K(z) = \int_0^z dw \mathcal{F}_K(w) \left(2m\sigma k^2 w (\phi^{(0)}(w) - \pi^{(0)}(w)) - k^2 \sigma^2 w^3 \pi^{(1)}(w) \right). \quad (\text{C16})$$

In terms of these definitions we can express the solution ϕ (C5, C11, C12, C13):

$$\begin{aligned} \phi(z) = & 1 - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} z I_{1/3}(k\sigma z^3) - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} z K_{1/3}(k\sigma z^3) \\ & + \mu_0 z I_{1/3}(k\sigma z^3) + \mu_I(z) z I_{1/3}(k\sigma z^3) + \mu_K(z) z K_{1/3}(k\sigma z^3) + \mathcal{O}(m^2). \end{aligned} \quad (\text{C17})$$

APPENDIX D: THE CANONICAL NORMALIZATION OF THE ϕ FIELD AND THE PARAMETER L_4 OF THE CHIRAL LAGRANGIAN

In this section we will present the explicit expressions for the quantities N_π and $g_m \phi^2$ introduced in the section **III** eq. (33).

Let us consider the quadratic part of the 5D action (6) where we only take into account the first Kaluza–Klein mode of the field ϕ which corresponds to the pion. We obtain:

$$S_{5D} \rightarrow \int dz \left(\Lambda^2 R^3 \frac{v^2(z)}{z^3} f_\phi^2(z) + \frac{R}{g_5^2} \frac{1}{z} \partial_z f_\phi^2(z) \right) \times \int d^4x \frac{1}{2} \partial_\mu \phi^a(x) \partial^\mu \phi^a(x). \quad (\text{D1})$$

Following (33) let us denote

$$\begin{aligned} \lim_{m \rightarrow 0} \int dz \left(\Lambda^2 R^3 \frac{v^2(z)}{z^3} f_\phi^2(z) + \frac{R}{g_5^2} \frac{1}{z} \partial_z f_\phi^2(z) \right) = & \int dz \left(\Lambda^2 R^3 \sigma^2 z^3 f_\phi^2(z) \Big|_{m=0} \right. \\ & \left. + \frac{R}{g_5^2} \frac{1}{z} \partial_z f_\phi^2(z) \Big|_{m=0} \right) = N_\pi^2 \end{aligned} \quad (\text{D2})$$

. We immediately obtain:

$$F_\pi \pi^a = N_\pi \phi^a, \quad (\text{D3})$$

where $f_\phi^2(z) \Big|_{m=0}$ may be extracted from (C17,E2).

The term of the Chiral Lagrangian

$$\mathcal{P}_4 = \text{tr}(D_\mu U^\dagger D^\mu U) \text{tr}(U^\dagger \chi + \chi^\dagger U) = 8 \frac{\Sigma}{F_\pi^2} m \partial_\mu \pi^a(x) \partial^\mu \pi^a(x) + \mathcal{O}(\pi^4) \quad (\text{D4})$$

from the AdS/QCD point of view is generated by the action (D1):

$$S_{5D} \rightarrow m \int d^4x \partial_\mu \phi^a(x) \partial^\mu \phi^a(x) \times \int dz \left(\frac{1}{2} \Lambda^2 R^3 \sigma z f_\phi^2(z) \Big|_{m=0} \right. \\ \left. + \Lambda^2 R^3 \sigma^2 z^3 f_\phi(z) \partial_m f_\phi(z) \Big|_{m=0} + \frac{R}{g_5^2 z} \partial_z f_\phi(z) \partial_m \partial_z f_\phi(z) \Big|_{m=0} \right). \quad (D5)$$

If we denote according to (33)

$$\int dz \left(\frac{1}{2} \Lambda^2 R^3 \sigma z f_\phi^2(z) \Big|_{m=0} + \Lambda^2 R^3 \sigma^2 z^3 f_\phi(z) \partial_m f_\phi(z) \Big|_{m=0} \right. \\ \left. + \frac{R}{g_5^2 z} \partial_z f_\phi(z) \partial_m \partial_z f_\phi(z) \Big|_{m=0} \right) = g_m \phi^2, \quad (D6)$$

we obtain:

$$L_4 = \frac{F_\pi^4}{8\Sigma} g_m \phi^2 N_\pi^{-2}, \quad (D7)$$

where $f_\phi^2(z) \Big|_{m=0}$ and $\partial_m f_\phi^2(z) \Big|_{m=0}$ may be extracted from (C7–C17, E2).

APPENDIX E: THE PARAMETERS L_1, L_2 AND L_3 OF THE CHIRAL LAGRANGIAN

In this section we will present the explicit expression for the quantity g_{ϕ^4} introduced in the section **III** eq. (33).

The ϕ^4 interaction (the Skyrme-like term in (33)) is generated by the quartic part of the gauge sector of the full 5D action (29):

$$S_{\phi^4} = \int d^5x \frac{-R}{4g_5^2 z} f^{abe} f^{cde} \partial_\mu \phi^a \partial_\nu \phi^b \partial^\mu \phi^c \partial^\nu \phi^d. \quad (E1)$$

The solution (C17) for $\phi(z)$ from subsection **C** is the function $f_\phi(z)$ with which we were dealing in section **III**, eq. (30). We will restrict our consideration to the chiral limit. This corresponds to the zero-order solution $\phi^{(0)}(z)$ (C5):

$$f_\phi(z) = N_\phi \left(1 - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} z I_{1/3}(k\sigma z^3) \right. \\ \left. - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} z K_{1/3}(k\sigma z^3) \right). \quad (E2)$$

The normalization factor N_ϕ is determined, as in (32), by the condition (31):

$$\int_\epsilon^{z_m} \frac{dz}{z} f_\phi^2(z) = 1. \quad (E3)$$

The effective 4D coupling is (E1, 30):

$$g_{\phi^4} = \int dz \frac{-R}{4g_5^2 z} f_\phi^4(z) = N_\phi^4 \int_\epsilon^{z_m} dz \frac{-R}{4g_5^2 z} \times \left(1 - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} \right. \\ \left. \times \frac{K_{-2/3}(k\sigma z_m^3)}{I_{-2/3}(k\sigma z_m^3)} z I_{1/3}(k\sigma z^3) - 2\Gamma^{-1}(1/3) \left(\frac{k\sigma}{2} \right)^{1/3} z K_{1/3}(k\sigma z^3) \right)^4. \quad (\text{E4})$$

We obtain an effective 4D Lagrangian for the pions:

$$\mathcal{L}_{\pi^4} = g_{\phi^4} \frac{F_\pi^4}{N_\pi^4} f^{abe} f^{cde} \partial_\mu \pi^a \partial_\nu \pi^b \partial^\mu \pi^c \partial^\nu \pi^d, \quad (\text{E5})$$

where N_π is the normalization factor that determines the proportionality between the fields π^a and ϕ^a , see the subsection **D**, eq. (D2).

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